Tutorial Notes 3

1. Evaluate
$$\int_0^2 \int_0^{4-x^2} \int_0^x \frac{\sin 2z}{4-z} \,\mathrm{d}y \,\mathrm{d}z \,\mathrm{d}x.$$

   Solutions:

   By Fubini's theorem,
$$\begin{aligned}
&\int_0^2 \int_0^{4-x^2} \int_0^x \frac{\sin 2z}{4-z} \,\mathrm{d}y \,\mathrm{d}z \,\mathrm{d}x \\
= &\int_0^2 \int_0^{4-x^2} x \frac{\sin 2z}{4-z} \,\mathrm{d}z \,\mathrm{d}x \\
= &\int_0^4 \int_0^{\sqrt{4-z}} x \frac{\sin 2z}{4-z} \,\mathrm{d}x \,\mathrm{d}z \\
= &\int_0^4 \frac{\sin 2z}{2} \,\mathrm{d}z \\
= &\frac{1-\cos 8}{4}.
\end{aligned}$$

2. Find the centroid of the region Ω: $0 \leq y \leq \mathrm{e}^x$, $x \leq 0$.

   Solutions:
$$\begin{aligned}
M &= \int_\Omega \mathrm{d}x \,\mathrm{d}y = \int_{-\infty}^0 \int_0^{\mathrm{e}^x} \mathrm{d}y \,\mathrm{d}x = \int_{-\infty}^0 \mathrm{e}^x \,\mathrm{d}x = 1; \\
M_x &= \int_\Omega x \,\mathrm{d}x \,\mathrm{d}y = \int_{-\infty}^0 \int_0^{\mathrm{e}^x} x \,\mathrm{d}y \,\mathrm{d}x = \int_{-\infty}^0 x\mathrm{e}^x \,\mathrm{d}x \\
&= x\mathrm{e}^x|_{-\infty}^0 - \int_{-\infty}^0 \mathrm{e}^x \,\mathrm{d}x = -1; \\
M_y &= \int_\Omega y \,\mathrm{d}x \,\mathrm{d}y = \int_{-\infty}^0 \int_0^{\mathrm{e}^x} y \,\mathrm{d}y \,\mathrm{d}x = \int_{-\infty}^0 \frac{\mathrm{e}^{2x}}{2} \,\mathrm{d}x = \frac{1}{4}.
\end{aligned}$$

   Hence $\bar{x} = -1$ and $\bar{y} = 1/4$.

3. Find the centroid of the region Ω: $z \geq 0$, $x^2 + 4y^2 \leq 4$, $z \leq 2 - x$.

   Solutions:

   Firstly,
$$M = \int_\Omega \mathrm{d}x \,\mathrm{d}y \,\mathrm{d}z = \int_{x^2+4y^2 \leq 4} \int_0^{2-x} \mathrm{d}z \,\mathrm{d}x \,\mathrm{d}y = \int_{x^2+4y^2 \leq 4} (2-x) \,\mathrm{d}x \,\mathrm{d}y.$$

By symmetry,

$$\int_{x^2+4y^2\le 4} x\,\mathrm{d}x\,\mathrm{d}y = 0.$$

Moreover,

$$\int_{x^2+4y^2\le 4} \mathrm{d}x\,\mathrm{d}y = \int_{x'^2+y'^2\le 1} 2\,\mathrm{d}x'\,\mathrm{d}y' = 2\pi. \tag{1}$$

Hence $M = 4\pi$.

$$\begin{aligned} M_x &= \int_\Omega x\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_{x^2+4y^2\le 4}\int_0^{2-x} x\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{x^2+4y^2\le 4} x(2-x)\,\mathrm{d}x\,\mathrm{d}y = -\int_{x^2+4y^2\le 4} x^2\,\mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

By symmetry,

$$\begin{aligned} \int_{x^2+4y^2\le 4} x^2\,\mathrm{d}x\,\mathrm{d}y &= \int_{x'^2+y'^2\le 1} 4x'^2\cdot 2\,\mathrm{d}x'\,\mathrm{d}y' = 8\int_{x^2+y^2\le 1} x^2\,\mathrm{d}x\,\mathrm{d}y \\ &= 4\int_{x^2+y^2\le 1}(x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y = 4\int_0^1\int_0^{2\pi} r^2\cdot r\,\mathrm{d}r\,\mathrm{d}\theta = 2\pi. \end{aligned} \tag{2}$$

Hence $M_x = -2\pi$. By symmetry,

$$M_y = \int_\Omega y\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_{x^2+4y^2\le 4}\int_0^{2-x} y\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y = \int_{x^2+4y^2\le 4} y(2-x)\,\mathrm{d}x\,\mathrm{d}y = 0.$$

Finally, by (1) and (2),

$$\begin{aligned} M_z &= \int_\Omega z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_{x^2+4y^2\le 4}\int_0^{2-x} z\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y = \int_{x^2+4y^2\le 4}\frac{(2-x)^2}{2} \\ &= \int_{x^2+4y^2\le 4}\frac{4-4x+x^2}{2}\,\mathrm{d}x\,\mathrm{d}y = 4\pi+\pi = 5\pi. \end{aligned}$$

Therefore, $\bar{x} = -1/2$, $\bar{y} = 0$, $\bar{z} = 5/4$.

4. Evaluate

$$\int_{\pi/6}^{\pi/3}\int_{\csc\phi}^{2\csc\phi}\int_0^{2\pi}\rho^2\sin\phi\,\mathrm{d}\theta\,\mathrm{d}\rho\,\mathrm{d}\phi.$$

Solutions:

$$\begin{aligned} &\int_{\pi/6}^{\pi/3}\int_{\csc\phi}^{2\csc\phi}\int_0^{2\pi}\rho^2\sin\phi\,\mathrm{d}\theta\,\mathrm{d}\rho\,\mathrm{d}\phi \\ &= 2\pi\int_{\pi/6}^{\pi/3}\int_{\csc\phi}^{2\csc\phi}\rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi \\ &= 2\pi\int_{\pi/6}^{\pi/3}\frac{7\csc^3\phi}{3}\sin\phi\,\mathrm{d}\phi \\ &= \frac{14\pi}{3}\int_{\pi/6}^{\pi/3}\frac{1}{\sin^2\phi}\,\mathrm{d}\phi \\ &= \frac{14\pi}{3}\left(-\cot\phi\right)\Big|_{\pi/6}^{\pi/3} \end{aligned}$$

$$= \frac{28\sqrt{3}}{9}.$$

5. Let D be the smaller cap cut from a solid ball of radius 2 units by a plane 1 unit from the center of the ball. Express the volume of D in Cartesian, cylindrical, spherical coordinates and evaluate the volume.

Solutions:

(a) Cartesian coordinates:

D is $x^2 + y^2 + z^2 \leq 4$, $1 \leq z \leq 2$. Then

$$\begin{aligned}
|D| &= \int_1^2 \int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} \int_{-\sqrt{4-y^2-z^2}}^{\sqrt{4-y^2-z^2}} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&= \int_1^2 \int_{x^2+y^2 \leq 4-z^2} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&= \int_1^2 \pi(4-z^2)\,\mathrm{d}z \\
&= \frac{5\pi}{3}.
\end{aligned}$$

(b) Cylindrical coordinates:

D is $r^2 + z^2 \leq 4$, $1 \leq z \leq 2$, $0 \leq \theta \leq 2\pi$. Then

$$\begin{aligned}
|D| &= \int_1^2 \int_0^{\sqrt{4-z^2}} \int_0^{2\pi} r\,\mathrm{d}\theta\,\mathrm{d}r\,\mathrm{d}z \\
&= 2\pi \int_1^2 \int_0^{\sqrt{4-z^2}} r\,\mathrm{d}r\,\mathrm{d}z \\
&= \pi \int_1^2 (4-z^2)\,\mathrm{d}z \\
&= \frac{5\pi}{3}.
\end{aligned}$$

(c) Spherical coordinates:

D is $0 \leq r \leq 2$, $1 \leq r\cos\phi \leq 2$. Hence $0 \leq \phi \leq \pi/3$. Then

$$\begin{aligned}
|D| &= \int_0^{\pi/3} \int_{\sec\phi}^2 \int_0^{2\pi} r^2 \sin\phi\,\mathrm{d}\theta\,\mathrm{d}r\,\mathrm{d}\phi \\
&= 2\pi \int_0^{\pi/3} \int_{\sec\phi}^2 r^2 \sin\phi\,\mathrm{d}r\,\mathrm{d}\phi \\
&= 2\pi \int_0^{\pi/3} \frac{8-\sec^3\phi}{3} \sin\phi\,\mathrm{d}\phi \\
&= 2\pi \left(\int_0^{\pi/3} \frac{8}{3} \sin\phi\,\mathrm{d}\phi - \int_0^{\pi/3} \frac{1}{3} \cdot \frac{\sin\phi}{\cos^3\phi}\,\mathrm{d}\phi \right)
\end{aligned}$$

$$
\begin{aligned}
&= 2\pi\left(\frac{4}{3} - \frac{1}{3}\int_0^{\pi/3} \frac{\mathrm{d}(-\cos\phi)}{\cos^3\phi}\right) \\
&= 2\pi\left(\frac{4}{3} - \frac{1}{3}\int_{1/2}^{1} \frac{1}{u^3}\,\mathrm{d}u\right) \\
&= \frac{5\pi}{3}.
\end{aligned}
$$